# Presburger Arithmetic an
# Pseudo-Recursive Saturati

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**Definition 1.** A model of **Presburger arithmeti**
an ordered abelian group with least positive eleme
the axiom schema

$$\forall x \in \Gamma \; \exists y \in \Gamma \; \exists i \in \{0, 1, \ldots, n-1\} \, (x = ny +$$

**Definition 2.** For $\Gamma$ a model of Presburger arithm
**ural residue map** $\varrho \colon \Gamma \to \widehat{\mathbb{Z}}$ is the homomorphism

$$\varrho(\gamma) = (\gamma \pmod 1, \gamma \pmod 2, \gamma \pmod 3$$

for all $\gamma \in \Gamma$.

**Definition 3.** For $a, b \in \Gamma$ with $a, b > \mathbb{Z}$, we define

$$\mathrm{st}\left(\frac{a}{b}\right) = \left\{\frac{n}{m} \in \mathbb{Q} : nb < ma\right\}.$$

This is an extended cut, identified with an exte
$[0, \infty] \subseteq \mathbb{R} \cup \{\infty\}$, where $r = \sup \mathrm{st}\left(\frac{a}{b}\right)$.

**Definition 4.** For a model $\Gamma$ the **set of standard**
is defined to be

$$\mathrm{stQ}(\Gamma) = \left\{\mathrm{st}\left(\frac{a}{b}\right) : a, b \in \Gamma\right\}.$$

**Lemma 5.** For $a, b, c \in \Gamma, q \in \mathbb{Q}$ the following hold

1. $\mathrm{st}\left(\frac{a}{b}\right) \cdot \mathrm{st}\left(\frac{b}{c}\right) = \mathrm{st}\left(\frac{a}{c}\right)$ provided the LHS is defir

2. $\mathrm{st}\left(\frac{qa}{b}\right) = q \cdot \mathrm{st}\left(\frac{a}{b}\right)$;

3. $\mathrm{st}\left(\frac{a}{qb}\right) = \frac{1}{q} \cdot \mathrm{st}\left(\frac{a}{b}\right)$ for $q \neq 0$;

4. $\mathrm{st}\left(\frac{a+b}{c}\right) = \mathrm{st}\left(\frac{a}{c}\right) + \mathrm{st}\left(\frac{b}{c}\right)$ provided the RHS is

5. if $a \leq b$ then $\mathrm{st}\left(\frac{a}{c}\right) \leq \mathrm{st}\left(\frac{b}{c}\right)$;

6. if $\mathrm{st}\left(\frac{a}{b}\right) \not\in \{0, \pm\infty\}$ then $\mathrm{st}\left(\frac{a}{b}\right) = \mathrm{st}\left(\frac{b}{a}\right)^{-1}$.

**Definition 6.** If $a, b \in \Gamma$ then $a \equiv b$ if either $a = b$
and

$$\mathrm{st}\left(\frac{a}{b}\right) \notin \{0, \pm\infty\}.$$

**Definition 7.** we call $\mathsf{V} = \Gamma/\equiv$ the set of **values**
a valuation map $\mathsf{v}\colon \Gamma \to \mathsf{V}$ by

$$\mathsf{v}\colon a \mapsto a/\equiv.$$

This valuation map is a reversal of the usual termi
$\mathsf{v}(\gamma_1 + \gamma_2) \leq \max\{\mathsf{v}(\gamma_1), \mathsf{v}(\gamma_2)\}$ .

**Definition 8.** A model of Presburger arithmetic
**recursively saturated** if $\Gamma \not\cong \mathbb{Z}$ and

1. for $\varrho \colon \Gamma \to \widehat{\mathbb{Z}}$ and each $r \in \mathrm{Im}(\varrho)$, the inverse in
   dense in $\Gamma$;

2. for $x, y, z \in \Gamma$ with $z \notin \mathbb{Z}$, there is some $w \notin \mathbb{Z}$ f

$$\mathrm{st}\left(\frac{w}{z}\right) = \mathrm{st}\left(\frac{x}{y}\right);$$

3. the set of values V is a dense linear order hav
   0 and no greatest point.

**Theorem 9.** Suppose $\Gamma$ is 2-homogeneous, then
are equivalent:

1. $\Gamma$ has no smallest non-standard value, and ther
   trivial $g \in G$;

2. there is some $x \in \Gamma$ with $\varrho(x) = 0$ and there are
   elements with value less than $\mathrm{v}(x)$;

3. there is a value-defying automorphism $h \in G$;

4. there exists a unique maximal convex submoc
   is pseudo-recursively saturated.

**Theorem 10.** If Γ is a countable pseudo-recursi
model then Γ is homogeneous.

**Example 11.**

$$G_{\mathsf{v}} = \{g \in G : \mathsf{v}(\gamma g) = \mathsf{v}(\gamma) \text{ for all } \gamma \in$$

is a non-trivial, proper, closed normal subgroup o

**Definition 12.** If $S_n \subseteq (\mathrm{stQ}(\Gamma)_{>0})^n \subseteq (\mathbb{R}^*_{>0})^n$ the
is stQ-**closed** if

1. each $S_n$ is nonempty and closed under pointw
   tion;

2. each $S_n$ is closed under pointwise inversion;

3. when $(r_1, \ldots, r_n) \in S$ and $m \leq n$ then

$$(r_1, \ldots, r_{m-1}, r_{m+$$

4. when $(r_1, \ldots, r_n) \in S$ and $m \leq n+1$ then there
   one $r'_m$ so that $(r_1, \ldots, r_{m-1}, r'_m, r_m, \ldots, r_n) \in$

**Definition 13.** If $S \subseteq \bigcup_{n\in\omega}(\mathsf{stQ}(\Gamma)_{>0})^n$ is stQ-cl
is the set of automorphisms

$$G_S = \Bigg\{ g \in G_{\mathsf{v}} : \forall n \in \omega \, \forall x_1, \ldots, x_n \in \Gamma$$

$$\left( \mathsf{st}\left(\frac{x_1 g}{x_1}\right), \ldots, \mathsf{st}\left(\frac{x}{}\right.\right.$$

**Theorem 14.** If $S \subseteq \bigcup_{n\in\omega}(\mathsf{stQ}(\Gamma)_{>0})^n$ is stQ-clo
a closed normal subgroup of $G$.

**Theorem 15.** Suppose that $G$ has trivial centre
a closed normal subgroup of $G$. If

$$S = \left\{ \left( \mathsf{st}\left(\frac{x_1 g}{x_1}\right), \ldots, \mathsf{st}\left(\frac{x_n g}{x_n}\right) \right) : n \in \omega, g \in N, x_1, \right.$$

then $N = G_S$.

**Proposition 16.** Suppose $T_1, T_2 \subseteq \bigcup_{n\in\omega}(\mathsf{stQ}(\Gamma)$
closed with $T_1 \subset T_2$. Then $G_{T_1} \subset G_{T_2}$.

**Definition 17.** If $S \subseteq \bigcup_{n\in\omega}(\mathsf{stQ}(\Gamma)_{>0})^n$ then we c
**reduction** of $S$ to be:

$$\overline{S}^{\mathsf{stQ}} = \bigcup_{\substack{T \subseteq \langle S \rangle \\ T \text{ stQ-closed}}} T.$$

**Proposition 18.** If $S \subseteq \bigcup_{n\in\omega}(\mathsf{stQ}(\Gamma))^n$ then the s
closed.

**Proposition 19.** Let $T_1$ and $T_2$ be stQ-closed. T

$$G_{\langle T_1 \cup T_2 \rangle} = \overline{\left\langle G_{T_1} \cup G_{T_2} \right\rangle}.$$

**Proposition 20.** Let $T_1$ and $T_2$ be stQ-closed. T

$$G_{\overline{T_1 \cap T_2}^{\mathsf{stQ}}} = G_{T_1} \cap G_{T_2}.$$

**Lemma 21.** Suppose $P = \{p_1, \ldots, p_n\}$ is a set
that $p \notin P$. Then $\langle P \rangle \neq \langle P \cup \{p\} \rangle$ where

$$\langle P \rangle = \{ s \in \mathbb{Q} : n \in \omega, x_1, \ldots, x_n \in P, l_1, \ldots, l_n \in \mathbb{Z}, s$$

**Corollary 22.** Let $\Gamma$ be a countable pseudo-recurs
model of Presburger arithmetic with trivial centre
$2^{\aleph_0}$ closed normal subgroups.

**Definition 23.** The set $B \subseteq \Gamma$ is **strongly indepe**
and every non-trivial $\mathbb{Q}$-linear combination

$$a = q_1 b_1 + \cdots + q_n b_n$$

has value

$$\mathrm{v}(a) = \max\{\, \mathrm{v}(b_j) : 1 \leq j \leq n, q_j \neq 0\}$$

where $q_1, \ldots, q_n \in \mathbb{Q}$ and $b_1, \ldots, b_n \in B$.

**Lemma 24 (Exchange Lemma).** If $a_1, \ldots, a_n$ a
dependent in $\Gamma$, and $a \in \Gamma$ then

either $a \in \langle a_1, \ldots, a_n \rangle$

or $\exists\, a_{n+1} \in \langle a_1, \ldots, a_n, a \rangle$ such that $a_1, \ldots, a$

are strongly independent and $a \in \Big\langle a_1, .$

**Lemma 25.** If $\{x_1, \dots, x_n\} \subseteq \Gamma$ and $\{y_1, \dots, y_n\}$
strongly independent sets, then the following are

1. $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{y})$;

2. $\varrho(x_i) = \varrho(y_i)$ and $\mathrm{st}\left(\frac{x_i}{x_j}\right) = \mathrm{st}\left(\frac{y_i}{y_j}\right)$ for all $1 \leq i$

*Proof.* Follows from quantifier elimination.

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